SUMMATION OF HYPERHARMONIC SERIES

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ABSTRACT. We shall show that the sum of the series formed by the so-called hyperharmonic numbers can be expressed in terms of the Riemann zeta function. More exactly, we give summation formula for the general hyperharmonic series.

1. HYPERHARMONIC NUMBERS

Introduction. In 1996, J. H. Conway and R. K. Guy in [CG] have defined the notion of hyperharmonic numbers.

The n-th harmonic number is the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

$H_n^{(1)} := H_n$, and for all $r > 1$ let

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}$$

be the n-th hyperharmonic number of order r. These numbers can be expressed by binomial coefficients and ordinary harmonic numbers:

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}). \tag{1}$$

It turned out that the hyperharmonic numbers have many combinatorial connections. To present this fact, we need to introduce the notion of r-Stirling numbers. Getting deeper insight, see [BGG] and the references given there.

r-Stirling numbers. $\left[{n \atop k}\right]_r$ is the number of permutations of the set $\{1, \ldots, n\}$ having k disjoint, non-empty cycles, in which the elements 1 through r are restricted to appear in different cycles.

The following identity integrates the hyperharmonic- and the r-Stirling numbers.

$$\frac{\left[{n+r \atop r+1}\right]_r}{n!} = H_n^{(r)}.$$

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This equality will be used in the special case $r = 1$ [GKP]:

$$\frac{\left[{n+1 \atop 2}\right]}{n!} := \frac{\left[{n+1 \atop 2}\right]_r}{n!} = H_n. \tag{2}$$

2. Results up to the present

Our goal is to determine the sum of the series has the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m},$$

for all $r \geq 2$ and possible m.

To do this, we should determine the asymptotic behaviour of hyperharmonic numbers. In the paper [M] there is a skimped approximation which helps us to get convergence theorems for hyperharmonic series. Namely, we have that

$$\frac{1}{r!} n^{r-1} < H_n^{(r)} < \frac{3}{2} \frac{(2r)^r}{(r-1)!} n^r,$$

for all $n \in \mathbb{N}$ and $r \geq 2$. According to this result, the followings were proved:

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^r} = \infty,$$

$$\frac{\zeta(s+1)}{r!} < \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^{r+s}} < \frac{3}{2} \frac{(2r)^r}{(r-1)!} \zeta(s) \quad (s > 1),$$

$$\sum_{n=1}^{\infty} \frac{n^{r-1}}{H_n^{(r)}} = \infty,$$

$$\frac{2}{3} \frac{(r-1)!}{(2r)^r} \zeta(s) < \sum_{n=1}^{\infty} \frac{n^{r-s}}{H_n^{(r)}} < r! \zeta(s-1) \quad (s > 2),$$

where ζ is the Riemann zeta function and $r \geq 2$.

3. Asymptotic approximation

To have the exact asymptotic behaviour of hyperharmonic numbers we need the following inequality from [CG].

$$\frac{1}{2(n+1)} + \ln(n) + \gamma < H_n < \frac{1}{2n} + \ln(n) + \gamma \quad (n \in \mathbb{N}), \tag{3}$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.

Now we formulate our first result.

Lemma 1. *For all $n \in \mathbb{N}$ and for a fixed order $r \geq 2$ we have*

$$H_n^{(r)} \sim \frac{1}{(r-1)!} \left(n^{r-1} \ln(n) \right),$$

that is, the quotient of the left and right hand side tends to 1.

Proof. The binomial coefficient in (1) has the form
$$\binom{n+r-1}{r-1} = \frac{(n+r-1)!}{(r-1)!n!} = \frac{1}{(r-1)!}(n+1)(n+2)\cdots(n+r-1).$$
It means that for a fixed order r
$$\binom{n+r-1}{r-1} \sim \frac{1}{(r-1)!}n^{r-1}.$$
For the convenience let us introduce the abbreviation $t := r-1$. We should estimate the factor $H_{n+t} - H_t$ in (1). According to (3), we get that
$$H_{n+t} - H_t < \frac{1}{2(n+t)} + \ln(n+t) + \gamma - \frac{1}{2(t+1)} - \ln(t) - \gamma.$$
Since $\frac{1}{2(n+t)} < \frac{1}{2(1+t)}$ and $\ln(n+t) - \ln(t) = \ln\left(\frac{n+t}{t}\right)$, so
$$H_{n+t} - H_t < \ln(n+t).$$
The lower estimation can be deduced as follows
$$H_{n+t} - H_t > \frac{1}{2(n+t+1)} + \ln(n+t) + \gamma - \frac{1}{2t} - \ln(t) - \gamma >$$
$$> \ln(n+t) - \ln(t) - \frac{1}{2} = \ln(n+t) - \ln(t\sqrt{e}).$$
From these we get that
$$\ln(n+t) - \ln(t\sqrt{e}) < H_{n+t} - H_t < \ln(n+t),$$
whence
$$1 - \frac{\ln(t\sqrt{e})}{\ln(n+t)} < \frac{H_{n+t} - H_t}{\ln(n+t)} < 1.$$
The limit of the left-hand side formula is 1 as n tends to infinity. Therefore (remember that $t = r-1$)
$$H_{n+r-1} - H_{r-1} \sim \ln(n+r-1) \sim \ln(n).$$
Collecting the results above we get the statement of the Lemma. □

Corollary 2. *The following series are convergent*
$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m} < +\infty,$$
whenever $m > r$.

Proof. Because of Lemma 1,
$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m} < C \sum_{n=1}^{\infty} \frac{n^{r-1}\ln(n)}{n^m} < C \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} < +\infty$$
under the assumption $m \geq r+1$. □

4. Generating functions, Euler sums and Hypergeometric Series

In this section we introduce the notions needed in the proof.

Generating functions. Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. Then the function

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

is called the generating function of $(a_n)_{n\in\mathbb{N}}$. If $a_n = H_n$ we get that (see [GKP, BGG])

$$\sum_{n=0}^{\infty} H_n z^n = -\frac{\ln(1-z)}{1-z},$$

and in general

$$\sum_{n=0}^{\infty} H_n^{(r)} z^n = -\frac{\ln(1-z)}{(1-z)^r}. \tag{4}$$

The generating function

$$\frac{1}{m!}\left(-\ln(1-z)\right)^m = \sum_{n=1}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{z^n}{n!} \tag{5}$$

can be found in [GKP, B].

The well known polylogarithm functions can also be considered as generating functions belong to $a_n = \frac{1}{n^k}$ (for a fixed k).

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \quad (k = 1, 2, \dots).$$

The last remarkable function needed by us is

$$\frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} z^n.$$

Euler sums. The general Euler sum is an infinite sum whose general term is a product of harmonic numbers divided by some power of n, see the comprehensive paper [FS]. The sum

$$\sum_{n=1}^{\infty} \frac{H_n}{n^m} = \frac{1}{2}(m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1)$$

was derived by Euler (see [BB] and the references given there). Related series were studied by De Doelder in [dD] and Shen [S], for instance.

Hypergeometric series. The Pochhammer symbol is defined by the formula

$$(x)_n = x(x+1)\cdots(x+n-1), \tag{6}$$

with special cases $(1)_n = n!$ and $(x)_1 = x$. The definition of the hypergeometric function (or hypergeometric series) is the following:

$$ {}_nF_m \left(\begin{array}{cccc} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_m \end{array} \middle| \, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_n)_k}{(b_1)_k (b_2)_k \cdots (b_m)_k} \frac{z^k}{k!}. $$

This function will appear in the sum of the hyperharmonic numbers.

We shall need one more statement.

Lemma 3. *We have*

$$ \int \frac{\ln(z)}{(1-z)z} dt = \mathrm{Li}_2(1-z) + \frac{1}{2}\ln^2(z), $$

and for all $2 \le r \in \mathbb{N}$

$$ \int \frac{\ln(z)}{(1-z)z^r} dz = \int \frac{\ln(z)}{(1-z)z^{r-1}} dz - \frac{\ln(z)}{(r-1)z^{r-1}} - \frac{1}{(r-1)^2 z^{r-1}}, $$

or, equivalently,

$$ \int \frac{\ln(z)}{(1-z)z^r} dz = \mathrm{Li}_2(1-z) + \frac{1}{2}\ln^2(z) - \sum_{k=1}^{r-1} \left(\frac{\ln(z)}{k z^k} + \frac{1}{k^2 z^k} \right). $$

up to additive constants

Proof. The definition of Li_2 readily gives that

$$ \mathrm{Li}_2'(1-z) = \frac{\ln(z)}{1-z}. $$

Moreover,

$$ \left[\frac{1}{2}\ln^2(z) \right]' = \frac{\ln(z)}{z}, $$

whence

$$ \mathrm{Li}_2'(1-z) + \left[\frac{1}{2}\ln^2(z) \right]' = \frac{z\ln(z) + (1-z)\ln(z)}{(1-z)z} = \frac{\ln(z)}{(1-z)z}. $$

The first statement is proved. The second one also can be deduced by differentiation. The derivative of the right-hand side has the form

$$ \frac{\ln(z)}{(1-z)z^{r-1}} - \frac{(r-1)z^{r-2} - (r-1)^2\ln(z)z^{r-2}}{(r-1)^2(z^{r-1})^2} - \frac{-(r-1)}{(r-1)^2 z^r} = $$

$$ = \frac{z\ln(z)}{(1-z)z^r} - \frac{z^r - (r-1)\ln(z)z^r}{(r-1)(z^r)^2} + \frac{1}{(r-1)z^r} = $$

$$ = \frac{z^{r+1}\ln(z)(r-1) - z^r(1-z) + (1-z)(r-1)\ln(z)z^r + z^r(1-z)}{(r-1)z^{2r}(1-z)} = $$

$$ = \frac{z\ln(z) + (1-z)\ln(z)}{z^r(1-z)} = \frac{\ln(z)}{z^r(1-z)}, $$

as we want. $\square$

5. The summation formula

For the sake of simplicity, we introduce the notations

$$S(r,m) := \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m},$$

and

$$B(k,m) := {}_{m+1}F_m \left(\begin{matrix} 1, & 1, & \dots, & 1, & k+1 \\ 2, & 2, & \dots, & 2 & \end{matrix} \middle| 1 \right).$$

After these introductory steps we are ready to prove the main theorem.

Theorem 4. *If $r \ge 2$ and $m \ge r+1$, then*

$$S(r,m) = S(1,m) + \sum_{k=1}^{r-1} \frac{1}{k} \left[S(k,m-1) - B(k,m) \right].$$

Proof. We begin with the generating function (4). Division with z and integration gives that

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n} z^n = -\int \frac{\ln(1-z)}{z(1-z)^r} dz.$$

An integral transformation gives that

$$-\int \frac{\ln(1-z)}{z(1-z)^r} dz = \int \frac{\ln(z)}{(1-z)z^r} dz =$$

$$= \mathrm{Li}_2(z) + \frac{1}{2} \ln^2(1-z) - \sum_{k=1}^{r-1} \left(\frac{\ln(1-z)}{k(1-z)^k} + \frac{1}{k^2(1-z)^k} \right).$$

According to (4) and (5) one can write

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n} z^n = \mathrm{Li}_2(z) + \sum_{n=1}^{\infty} \begin{bmatrix} n \\ 2 \end{bmatrix} \frac{z^n}{n!} -$$

$$\sum_{k=1}^{r-1} \left(\frac{1}{k}(-1) \sum_{n=0}^{\infty} H_n^{(r)} z^n + \frac{1}{k^2} \sum_{n=0}^{\infty} \binom{n+k-1}{n} z^n \right).$$

It is obvious that this series is divergent but the generating functions works well without any restriction.

Let us deal with the second term. The Stirling numbers satisfy the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \quad (n > 0).$$

From this

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n \begin{bmatrix} n \\ 2 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} = n \begin{bmatrix} n \\ 2 \end{bmatrix} + (n-1)! \quad (n > 0).$$

Now, (2) can be rewritten as follows

$$H_n = \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = \frac{1}{(n-1)!}\begin{bmatrix} n \\ 2 \end{bmatrix} + \frac{1}{n}.$$

Division with n and rearrangement give that

$$\frac{1}{n!}\begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{H_n}{n} - \frac{1}{n^2}.$$

Therefore the second sum is

$$\sum_{n=1}^{\infty}\begin{bmatrix} n \\ 2 \end{bmatrix}\frac{z^n}{n!} = \sum_{n=1}^{\infty}\frac{H_n}{n}z^n - \sum_{n=1}^{\infty}\frac{z^n}{n^2}.$$

Since the last member equals to $\mathrm{Li}_2(z)$, it cancels the first member of the sum above. Hence

$$\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n}z^n = \sum_{n=1}^{\infty}\frac{H_n}{n}z^n + \sum_{k=1}^{r-1}\left(\frac{1}{k}\sum_{n=0}^{\infty}H_n^{(r)}z^n - \frac{1}{k^2}\sum_{n=0}^{\infty}\binom{n+k-1}{n}z^n\right).$$

An easy induction shows that (after dividing with z, integrating, and repeating these steps $(m-1)$-times and finally substituting $z=1$)

(7)

$$\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n^m} = S(1,m) + \sum_{k=1}^{r-1}\left(\frac{1}{k}S(r,m-1) - \frac{1}{k^2}\sum_{n=1}^{\infty}\binom{n+k-1}{n}\frac{1}{n^{m-1}}\right).$$

The last step is the transformation of the last member.

$$\sum_{n=1}^{\infty}\binom{n+k-1}{n}\frac{1}{n^{m-1}} = \sum_{n=1}^{\infty}\frac{(n+k-1)!}{n!(k-1)!}\frac{1}{n^{m-1}} =$$

$$\frac{1}{(k-1)!}\sum_{n=1}^{\infty}(n+1)(n+2)\cdots(n+k-1)\frac{1}{n^{m-1}} =$$

$$\frac{1}{(k-1)!}\sum_{n=1}^{\infty}\frac{(n)_k}{n^m},$$

because of the definition of the Pochhammer symbol in formula (6). On the other hand, the definition of $B(k,m)$ yields that

$$B(k,m) = \sum_{n=0}^{\infty}\frac{(n!)^m}{(n+1)!^m}\frac{(k+1)_n}{n!} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^m}\frac{(k+1)_n}{n!}.$$

The next conversion should be applied:

$$\frac{k!(k+1)_n}{n!} = \frac{(k+n)!}{n!} = (n+1)(n+2)\cdots(n+k) = (n+1)_k.$$

It means that the equality

$$B(k,m) = \frac{1}{k!}\sum_{n=0}^{\infty}\frac{(n+1)_k}{(n+1)^m} = \frac{1}{k!}\sum_{n=1}^{\infty}\frac{(n)_k}{n^m} \tag{8}$$

holds. That is,

$$\sum_{n=1}^{\infty} \binom{n+k-1}{n} \frac{1}{n^{m-1}} = kB(k, m).$$

Considering this and (7) the result follows. □

6. Tabular of the low-order sums

In the following tabulars we collect the low-order results of the Summation Theorem. We used the following identities which can be easily derived from (8) and (6).

$$B(1, m) = \zeta(m-1),$$
$$B(2, m) = \frac{1}{2}\left(\zeta(m-1) + \zeta(m-2)\right),$$
$$B(3, m) = \frac{1}{6}\zeta(m-3) + \frac{1}{2}\zeta(m-2) + \frac{1}{3}\zeta(m-1).$$

A computation with the mathematical package Maple shows that an improved accuracy can be reached using Theorem 1, in spite of calculating the series term-by-term. Namely, the sum

$$\sum_{n=1}^{10^5} \frac{H_n^{(4)}}{n^5} = 1.310972037,$$

and takes about 500 seconds on an average personal computer, while the "closed form" shown below gives immediately the closer value

$$\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^5} = \frac{\pi^6}{540} - \frac{\pi^4}{810} - \frac{11\pi^2}{216} - \zeta(3) - \frac{11\pi^2}{36}\zeta(3) - \frac{1}{2}\zeta(3)^2 + \frac{11}{2}\zeta(5)$$
$$\approx 1.310990854,$$

with ten digits accuracy.

$$S(2, m)$$

Power of n	Closed form	Approx. value
$m=3$	$\frac{\pi^4}{72} - \frac{\pi^2}{6} + 2\zeta(3)$	2.112083781
$m=4$	$\frac{\pi^4}{72} + 3\zeta(5) - \zeta(3)\left(1 + \frac{\pi^2}{6}\right)$	1.284326055
$m=5$	$\frac{\pi^6}{540} - \frac{\pi^4}{90} - \frac{1}{2}\zeta(3)^2 + 3\zeta(5) - \frac{\pi^2}{6}\zeta(3)$	1.109035642
$m=6$	$\frac{\pi^6}{540} + 4\zeta(7) - \frac{\pi^4}{90}\zeta(3) - \frac{1}{2}\zeta(3)^2 - \zeta(5)\left(1 + \frac{\pi^2}{6}\right)$	1.047657410
$m=7$	$\frac{\pi^8}{4200} - \frac{\pi^6}{945} - \zeta(5)\zeta(3) + 4\zeta(7) - \frac{\pi^2}{6}\zeta(5) - \frac{\pi^4}{90}\zeta(3)$	1.022090029
$m=8$	$\frac{\pi^8}{4200} + 5\zeta(9) - \frac{\pi^6}{945}\zeta(3) - \frac{\pi^4}{90}\zeta(5) - \zeta(5)\zeta(3) - \zeta(7)\left(1 + \frac{\pi^2}{6}\right)$	1.010557246

$m=9$	$\frac{\pi^{10}}{34020} - \frac{\pi^8}{9450} - \zeta(7)\zeta(3) - \frac{1}{2}\zeta(5)^2 + 5\zeta(9) - \frac{\pi^2}{6}\zeta(7) - \frac{\pi^6}{945}\zeta(3) - \frac{\pi^4}{90}\zeta(5)$	1.005133570
$m=10$	$\frac{\pi^{10}}{34020} + 6\zeta(11) - \frac{\pi^8}{9450}\zeta(3) - \frac{\pi^6}{945}\zeta(5) - \frac{1}{2}\zeta(5)^2 - \frac{\pi^4}{90}\zeta(7) - \zeta(7)\zeta(3) - \zeta(9)\left(1 + \frac{\pi^2}{6}\right)$	1.002522063

$S(3,m)$

Power of n	Closed form	Approx. value
$m=4$	$\frac{\pi^4}{48} - \frac{\pi^2}{8} - \frac{\pi^2}{6}\zeta(3) - \frac{1}{4}\zeta(3) + 3\zeta(5)$	1.628620203
$m=5$	$\frac{\pi^6}{540} - \frac{\pi^4}{144} - \frac{\pi^2}{4}\zeta(3) - \frac{3}{4}\zeta(3) - \frac{1}{2}\zeta(3)^2 + \frac{9}{2}\zeta(5)$	1.180103635
$m=6$	$\frac{\pi^6}{360} - \frac{\pi^4}{120} + 4\zeta(7) - \frac{\pi^2}{6}\zeta(5) - \frac{\pi^4}{90}\zeta(3) - \frac{3}{4}\zeta(3)^2 + \frac{1}{4}\zeta(5) - \frac{\pi^2}{12}\zeta(3)$	1.072362484
$m=7$	$\frac{\pi^8}{4200} - \frac{\pi^6}{2520} - \frac{\pi^4}{60}\zeta(3) - \frac{1}{4}\zeta(3)^2 - \zeta(5)\zeta(3) - \zeta(5)\left(\frac{\pi^2}{4} + \frac{3}{4}\right) + 6\zeta(7)$	1.032351029
$m=8$	$\frac{\pi^8}{2800} - \frac{\pi^6}{1260} - \zeta(3)\left(\frac{\pi^4}{180} + \frac{\pi^6}{945}\right) - \zeta(5)\left(\frac{\pi^2}{12} + \frac{\pi^4}{90}\right) - \frac{3}{2}\zeta(5)\zeta(3) + \zeta(7)\left(\frac{3}{4} - \frac{\pi^2}{6}\right) + 5\zeta(9)$	1.015179175

$S(4,m)$

Power of n	Closed form	Approx. value
$m=5$	$\frac{\pi^6}{540} - \frac{\pi^4}{810} - \frac{11\pi^2}{216} - \zeta(3) - \frac{11\pi^2}{36}\zeta(3) - \frac{1}{2}\zeta(3)^2 + \frac{11}{2}\zeta(5)$	1.310990854
$m=6$	$\frac{11\pi^6}{3240} - \frac{\pi^4}{80} - \zeta(3)\left(\frac{\pi^4}{90} + \frac{1\pi^2}{6} + \frac{11}{36}\right) + + \frac{11}{12}\zeta(3)^2 + \zeta(5)\left(\frac{59}{36} - \frac{\pi^2}{6}\right) + 4\zeta(7)$	1.103348021
$m=7$	$\frac{\pi^8}{4200} - \frac{11\pi^4}{3240} + \frac{\pi^6}{2430} - \frac{1}{2}\zeta(3)^2 - \zeta(3)\left(\frac{11\pi^4}{540} - \frac{\pi^2}{36}\right) - \zeta(3)\zeta(5) - \zeta(5)\left(\frac{5}{6} - \frac{11\pi^2}{36}\right) + \frac{22}{3}\zeta(7)$	1.043816710

$m=8$	$\frac{11\pi^8}{25200} - \frac{5\pi^6}{4536} - \zeta(3)\left(\frac{\pi^6}{945}+\frac{\pi^4}{90}\right) -$ $\frac{1}{12}\zeta(3)^2 - \frac{11}{6}\zeta(3)\zeta(5) -$ $\zeta(5)\left(\frac{\pi^4}{90}+\frac{\pi^2}{6}+\frac{11}{36}\right) +$ $\zeta(7)\left(\frac{95}{36}-\frac{\pi^2}{6}\right)+5\zeta(9)$	1.020093103

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